On subgroups of finite index in branch groups

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Abstract

We give a structural description of the normal subgroups of subgroups of finite index in branch groups in terms of rigid stabilizers. This gives further insight into the structure lattices of branch groups introduced by the second author. We derive a condition concerning abstract commensurability of branch groups acting on the p-ary tree for any prime p.

1 Introduction

The class of branch groups was introduced by Grigorchuk in 1997, to provide a general framework for studying groups arising as counter-examples in a wide variety of contexts. This class also plays a natural role in the study of just infinite groups (see [9]). By now, the structure theory of branch groups is quite well developed; see for example [1, 3, 9]. Among the remarkable properties of arbitrary branch groups is a result proved by Grigorchuk describing their non-trivial normal subgroups; it has the consequence that every proper quotient of a branch group is virtually abelian. The definition of branch groups (given below) shows that some subgroups of finite index (for example, subgroups $\mathrm{rist}_G(n)$ with $n > 0$) do not share this property. However, our main theorem, Theorem 1.2 below, gives a reasonably precise description of normal subgroups of subgroups of finite index in branch groups.

Branch groups are defined in terms of their action on a specific type of tree. Let $(m_n)_{n \geqslant 0}$ be a sequence of integers with $m_n \geqslant 2$ for each n. The rooted tree of type (m_n) is a tree T with a vertex v_0 (called the root vertex) of valency m_0, such that every vertex at a distance $n \geqslant 1$ from v_0 has valency $m_n + 1$. We picture T with the root at the top and with m_n edges descending from each vertex of level n. Therefore we call the vertices below a vertex v the *descendants* of v. If $m_n = d$ for every n, we say that T is a d-ary tree. The distance of a vertex v from v_0 is called the *level* of v, and the set L_n of vertices of level n is called the nth *layer* of T. Each vertex v of level r is the root of a rooted subtree T_v of type $(m_n)_{n \geqslant r}$.

Now suppose that G is a group of tree automorphisms of T fixing v_0. For each vertex v write $\mathrm{rist}_G(v)$ for the subgroup of elements of G that fix all vertices outside T_v, and for each $n \geqslant 0$ write $\mathrm{rist}_G(n)$ for the direct product $\langle \mathrm{rist}_G(v) \mid v \in L_n \rangle$. We also write $\mathrm{rist}_G(X) = \prod_{x \in X} \mathrm{rist}_G(x)$ for each subset

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X of L_n. The group G is said to be a *branch group* on T if the following two conditions hold for each $n \geqslant 0$:

(i) G acts transitively on L_n;

(ii) $|G : \mathrm{rist}_G(n)|$ is finite.

Branch groups are subject to strong restrictions. The proof of [4, Lemma 2] shows that G has no non-trivial virtually abelian normal subgroups and in [3, Theorem 4] the following description of normal subgroups is given:

Theorem 1.1. *Suppose that G is a branch group acting on a tree T and let $K \triangleleft G$ with $K \neq 1$. Then $\mathrm{rist}_G(n)' \leqslant K$ for some n.*

We will prove the following result for normal subgroups of subgroups of finite index:

Theorem 1.2. *Suppose that G is a branch group acting on a tree T, let H be a subgroup of finite index and let $K \triangleleft H$. Then for all sufficiently large integers n there is a union X of H-orbits in L_n such that*

$$K \cap \mathrm{rist}_G(n)' = \mathrm{rist}_G(X)'. \tag{1}$$

More precisely,

$$\mathrm{rist}_G(X)' \leqslant K \quad and \quad K \cap \mathrm{rist}_G(L_n \setminus X) = [K, \mathrm{rist}_G(L_n \setminus X)] = 1. \tag{2}$$

The second part of (2) above is simply the statement that the subgroups K and $\mathrm{rist}_G(L_n \setminus X)$ generate their direct product in G.

It will follow easily from this result that for any subgroup H of finite index of a branch group, the number of infinite H-invariant direct factors of a normal subgroup of H is bounded by $|G : H|$. These results allow us to establish in Section 3 a necessary condition for two direct products of branch groups acting on the p-ary tree to be commensurable, based on the number of direct factors. This complements a result of the first author [2] concerning the Gupta–Sidki 3-group.

We sketch another application. The structure lattice $\mathcal{L}$ and structure graph $\mathcal{T}$ of a branch group G have been defined and studied in [4, 9]. They depend only on the structure of G as an abstract group and not on its action on the tree T. We have $\mathcal{T} \subset \mathcal{L}$, and in some important cases $\mathcal{T}$ turns out to be isomorphic to T. Theorem 1.2 gives another approach to these objects, by associating unions of orbits in T with the subgroups K in the statement. Let $\mathcal{L}_0$ be the family of subgroups of the form $\mathrm{rist}_G(X)$ with X an H-invariant subset of a layer of T for some subgroup $H \leqslant_{\mathrm{f}} G$. For such subsets X_1, X_2 write $\mathrm{rist}_G(X_1) \sim \mathrm{rist}_G(X_2)$ if $X_2 \subseteq L_n$ consists of all descendants of X_1 in L_n (or vice-versa). Then $\sim$ is an equivalence relation on $\mathcal{L}_0$ and the quotient set with naturally defined operations is a Boolean lattice isomorphic to $\mathcal{L}$. Moreover $\mathcal{T}$ corresponds to the equivalence classes containing subgroups $\mathrm{rist}_G(X)$ for orbits X that do not split into unions of distinct H-orbits in lower layers. We leave the details to the interested reader.

2 Preliminaries and proof of Theorem 1.2

Notation. Throughout, we will write $H \leqslant_f G$ (resp. $H \vartriangleleft_f G$) to mean that H is a subgroup (resp. a normal subgroup) of finite index in G, and K^S will denote the subgroup of G generated by all conjugates of a subgroup K under a subset $S \subseteq G$.

We begin by noting an immediate consequence of the definition of branch groups. It will be used throughout the rest of the paper.

Remark 2.1. Let G be a branch group on a tree T with layers L_n. Since G acts transitively on each L_n, a subgroup $H \leqslant G$ of finite index will have at most $|G : H|$ orbits on L_n. If X is an H-orbit in L_n then the descendants of X on L_{n+1} comprise a union of H-orbits. Hence there is some n_0 such that the number of H-orbits on L_n is equal to the number of H-orbits on L_{n_0} for all $n \geqslant n_0$. Further, for each H-orbit in L_n ($n \geqslant n_0$) the set of its descendants in any lower layer is again an H-orbit.

The following preliminary results concern virtually soluble subgroups and factors of branch groups and will be used in the proof of Theorem 1.2. The next lemma shows some consequences of a result from [4].

Lemma 2.2. *Let G be a branch group on a tree T. Then G has no non-trivial abelian subgroups K satisfying $K \vartriangleleft K^G \vartriangleleft G$. Moreover, the following assertions hold*:

(a) $\mathrm{rist}_G(v)$ *is not virtually soluble, for each vertex v in T*;

(b) *if $H \leqslant_f G$ then H has no virtually soluble normal subgroups.*

Proof. The first assertion is [4, Lemma 2].

(a) The subgroup $\mathrm{rist}_G(v)$ is infinite since the direct product of its finitely many conjugates is some subgroup $\mathrm{rist}_G(n)$ and this has finite index in G. Thus if $\mathrm{rist}_G(v)$ is virtually soluble, then it must have a non-trivial abelian normal subgroup K. Since then $K \vartriangleleft K^G \vartriangleleft G$ this gives a contradiction.

(b) Now let $H \leqslant_f G$. First we claim that if K is a finite normal subgroup of H then $K = 1$. Let $C = \mathrm{C}_H(K)$; then, since $\mathrm{N}_H(K)/C \cong \mathrm{Aut}_H(K)$, we have $C \leqslant_f G$ and so $D \vartriangleleft_f G$ where $D = \bigcap_{g \in G} C^g$. Moreover D centralizes K^G and so $D \cap K^G$ is an abelian normal subgroup of G. From above $D \cap K^G = 1$ and so K^G is finite. If $K \neq 1$ then by Theorem 1.1 we have $\mathrm{rist}_G(n_1)' \leqslant K^G$ for some n_1. Because G is residually finite, there is some $L \vartriangleleft_f G$ with $L \cap K^G = 1$, and we have $\mathrm{rist}_G(n_2)' \leqslant L$ for some n_2. But then $\mathrm{rist}_G(n)$ must be abelian where $n = \max(n_1, n_2)$, and this is a contradiction.

Finally suppose that $1 \neq K \vartriangleleft H$ and that K is virtually soluble. The soluble normal subgroup K_0 of K of smallest index is normal in H and non-trivial from above, and the last non-trivial term A of the derived series of K_0 is an abelian normal subgroup of H. From above, A is infinite and so $A \cap M$ is a non-trivial abelian normal subgroup of M where $M = \bigcap_{g \in G} H^g$. But this gives another contradiction to the first assertion of the lemma. $\square$

Our next result is a variant on an argument in [8]. In our application of it we take for $\mathfrak{X}$ the class of virtually soluble groups. This is evidently quotient-closed and it is also extension-closed.

Lemma 2.3. *Let $\mathfrak{X}$ be a class of groups that is closed for quotients and extensions, and let $H \vartriangleleft_{\mathrm{f}} G$. If each ascending chain $(K_i)_{i \in \mathbb{N}}$ of normal subgroups of G has at most c factors K_i/K_{i-1} that are not in $\mathfrak{X}$, then each ascending chain of normal subgroups of H has at most $c \cdot 2^{|G:H|-1}$ factors that are not in $\mathfrak{X}$.*

Proof. We begin by noting that if A_1, A_2, B_1, B_2 are normal subgroups of H with $A_1 \leqslant A_2$ and $B_1 \leqslant B_2$, and if both $A_2 B_2 / A_1 B_1$, $(A_2 \cap B_2)/(A_1 \cap B_1)$ are in $\mathfrak{X}$, then A_2/A_1 is in $\mathfrak{X}$.

Write $n = |G : H|$. For the purposes of this proof, we call a non-empty finite subset S of G *good* if whenever (K_i) is an ascending chain of normal subgroups of H then the chain (K_i^S) has at most $c \cdot 2^{n-|S|}$ factors that are not in $\mathfrak{X}$. Thus if S is a transversal to H in G then S is good, and our conclusion holds if and only if $\{1\}$ is good. Choose a good set S of smallest cardinality. Then each set $S s_0^{-1}$ with $s_0 \in S$ is good, and so we may assume that $1 \in S$. Suppose that $S \neq \{1\}$ and write $S' = S \setminus \{1\}$; thus $S' \neq \emptyset$. Let (K_i) be an ascending chain of normal subgroups of H and write $K_i^* = K_i \cap K_i^{S'}$ for each i. Since S is good, the set J of indices j for which either K_j^S / K_{j-1}^S or K_j^{*S}/K_{j-1}^{*S} is not in $\mathfrak{X}$ has at most $2(c \cdot 2^{n-|S|}) = c \cdot 2^{n-|S'|}$ elements. However it is easy to check that

$$K_i^{S'}(K_i^{* \, S'} K_i) = K_i^S \quad \text{and} \quad K_i^{S'} \cap (K_i^{* \, S'} K_i) = K_i^{* \, S}$$

for each i. Therefore, for the indices $i \notin J$, the note in the first paragraph yields that $K_i^{S'}/K_{i-1}^{S'} \in \mathfrak{X}$. But this shows that the set S' is good, and the result follows from this contradiction. $\square$

This lemma has the following consequence which will be necessary for the proof of Theorem 1.2.

Lemma 2.4. *Let G be a branch group acting on a tree T, and $H \vartriangleleft_{\mathrm{f}} G$. Then in any series of normal subgroups of H there are at most $2^{|G:H|-1}$ factors that are not virtually soluble.*

Proof. By Theorem 1.1, every proper quotient of G is virtually abelian. Therefore the result holds from Lemma 2.3, with $c = 1$ and $\mathfrak{X}$ the class of virtually soluble groups. $\square$

We are now ready to prove the main theorem. For the reader's convenience, we present the easier part of it separately.

Lemma 2.5. *Let G be a branch group acting on a tree T, and $K \vartriangleleft H \leqslant_{\mathrm{f}} G$. If $K \cap \mathrm{rist}_G(v) > 1$ for all vertices v in T then $\mathrm{rist}_G(n)' \leqslant K$ for some n.*

Proof. Choose n_0 as in Remark 2.1 and also with $\mathrm{rist}_G(n_0)' \leqslant H$. Let $X_1, \ldots, X_r$ be the orbits of H on L_{n_0}.

Fix i and let v be a vertex in X_i. Choose $k \in K$ and $m_i \geqslant 0$ such that k does not fix the m_ith layer in T_v; choose u in this layer with $uk \neq u$. We claim that $\mathrm{rist}_G(u)' \leqslant K$. To see this, let $r_1, r_2 \in \mathrm{rist}_G(u)$; thus r_1^k lies in $\mathrm{rist}_G(uk)$ and acts trivially on T_u. Hence for all $w \in T_u$ we have $w[r_1^{-1}, k]^{\pm 1} = wr_1^{\pm 1}$, so

$$w[[r_1^{-1}, k], r_2] = w([r_1^{-1}, k]^{-1} r_2^{-1} [r_1^{-1}, k] r_2) = w[r_1, r_2].$$

Since $[[r_1^{-1}, k], r_2] \in K \cap \mathrm{rist}_G(u)$ we obtain that $[r_1, r_2] = [[r_1^{-1}, k], r_2] \in K$ and our claim follows. Since H normalizes K it follows that K contains $\mathrm{rist}_G(Y)'$ where Y is the H-orbit in $L_{n_0 + m_i}$ containing u. Arguing thus for each i, we conclude that $\mathrm{rist}_G(n)' \leqslant K$ where $n = n_0 + \max(m_1, \ldots, m_r)$. $\qquad\square$

Proof of Theorem 1.2. We begin by noting that the statements in (2) imply equation (1). If (2) holds then we have

$$\begin{aligned} K \cap \mathrm{rist}_G(n)' &= K \cap (\mathrm{rist}_G(X)' \times \mathrm{rist}_G(L_n \setminus X)') \\ &= \mathrm{rist}_G(X)' \times (K \cap \mathrm{rist}_G(L_n \setminus X)'). \end{aligned}$$

From (2), the second factor here is an abelian normal subgroup of H, and so is trivial by Lemma 2.2.

Therefore it suffices to prove (2). For each normal subgroup K of H, every chain of normal subgroups of H from K to H with no virtually soluble factors has length bounded independently of K, by Lemma 2.4. Write $j(K)$ for the maximal length of such a chain.

Suppose that the result is false. Pick a subgroup K demonstrating this and with $j(K)$ as small as possible. By Lemma 2.5 we have $\mathrm{rist}_G(v) \cap K = 1$ for some vertex v. Since H contains a normal subgroup of finite index, it contains $\mathrm{rist}_G(n_1)'$ for some $n_1 \in \mathbb{N}$. Let w be a vertex in T_v with $w \in L_{n_2}$ where $n_2 \geqslant n_1$. Let X_1 be the H-orbit of w.

The subgroups $\mathrm{rist}_G(w)'$ and $K \cap \mathrm{rist}_G(n_2)'$ are disjoint and both normal in $\mathrm{rist}_G(n_2)$, and hence they commute. But $K \triangleleft H$; so $\mathrm{rist}_G(u)'$ and $K \cap \mathrm{rist}_G(n_2)'$ commute for all $u \in X_1$, and hence $\mathrm{rist}_G(X_1)'$ and $K \cap \mathrm{rist}_G(n_2)'$ commute. Therefore K and $\mathrm{rist}_G(X_1)'$ intersect in a virtually abelian normal subgroup of H, so intersect trivially and commute by Lemma 2.2. It follows that these subgroups generate their direct product K_1. Since $K_1 \triangleleft H$ and $K_1/K \cong \mathrm{rist}_G(X_1)'$, which is not virtually soluble by Lemma 2.2 (a), we have $j(K_1) < j(K)$. Therefore we can find some $n_0 \geqslant n_2$ such that for every $n \geqslant n_0$ there is a union X_2 of H-orbits of L_n satisfying

$$\mathrm{rist}_G(X_2)' \leqslant K \times \mathrm{rist}_G(X_1)' \quad \text{and} \quad (K \times (\mathrm{rist}_G(X_1)') \cap \mathrm{rist}_G(L_n \setminus X_2) = 1. \quad (*)$$

Let Y_1 be the set of descendants of X_1 in L_n; thus $\mathrm{rist}_G(Y_1)' \leqslant \mathrm{rist}_G(X_1)'$ and hence $Y_1 \subseteq X_2$. Let $X = X_2 \setminus Y_1$; this is a union of H-orbits of L_n. The first inequality in $(*)$ gives

$$\mathrm{rist}_G(X)' \leqslant \mathrm{rist}_G(Y_1)' \times K,$$

and since $\mathrm{rist}_G(X)$, $\mathrm{rist}_G(Y_1)$ commute it follows that

$$(\mathrm{rist}_G(X))'' \leqslant [K \times \mathrm{rist}_G(Y_1)', \mathrm{rist}_G(X)] \leqslant [K, \mathrm{rist}_G(X)] \leqslant K.$$

Thus the image of $\mathrm{rist}_G(X)'$ under the projection map $\mathrm{rist}_G(Y_1)' \times K \to \mathrm{rist}_G(Y_1)'$ is abelian and H-invariant, and so is trivial from Lemma 2.2. Hence $\mathrm{rist}_G(X)' \leqslant K$. The second inequality in $(*)$ gives $K \cap \mathrm{rist}_G(L_n \setminus X_2) = 1$; thus $K \cap \mathrm{rist}_G(n)$ commutes with $\mathrm{rist}_G(L_n \setminus X_2)$ as well as $\mathrm{rist}_G(Y_1)$, and so commutes with $\mathrm{rist}_G(L_n \setminus X)$. The result follows. $\qquad\square$

We can now strengthen Lemma 2.4.

Corollary 2.6. *Let G be a branch group acting on a tree T, and $H \leqslant_{\mathrm{f}} G$. Then in any series of normal subgroups of H there are at most $|G : H|$ factors that are not virtually abelian.*

Proof. For an H-invariant subset X of a layer L_n of T we write $o(X)$ for the number of orbits of H in X.

Let $(K_i)_{i \in \mathbb{N}}$ be an ascending series of normal subgroups of H. Then, by Theorem 1.2, for each i there exist some n_i and some union X_i of H-orbits in L_{n_i} such that $\mathrm{rist}_G(n_i)' \cap K_i = \mathrm{rist}_G(X_i)'$. Moreover n_i can be chosen so that $o(X_i)$ is as large as possible (by Remark 2.1). Clearly $o(X_{i-1}) \leqslant o(X_i)$ for each i. Thus there are at most $|G : H|$ indices i for which $o(X_{i-1}) < o(X_i)$. Suppose that $j \in \mathbb{N}$ is not one of these indices. We claim that the quotient K_j/K_{j-1} is virtually abelian. To see this, let $n \geqslant n_{j-1}, n_j$, replace X_{j-1} and X_j by their respective sets of descendants in L_n, and notice that

$$\mathrm{rist}_G(X_{j-1})' = \mathrm{rist}_G(n)' \cap K_{j-1} \leqslant \mathrm{rist}_G(n)' \cap K_j = \mathrm{rist}(X_j)'.$$

Since $o(X_{j-1}) = o(X_j)$, equality holds above. By Theorem 1.1, the quotient $K_j/\mathrm{rist}_G(X_j)' \cong (K_j \cdot \mathrm{rist}_G(n)')/\mathrm{rist}_G(n)'$ is virtually abelian and therefore so is

$$\frac{K_j}{K_{j-1}} \cong \frac{K_j/\mathrm{rist}_G(X_j)'}{K_{j-1}/\mathrm{rist}_G(X_j)'}.$$

Our claim follows, and the result is proved. $\qquad\square$

3 An application: abstract commensurability

We recall that two groups G_1, G_2 are said to be abstractly commensurable if they have isomorphic subgroups of finite index. Let Γ be the Gupta–Sidki 3-group. This is a branch group on the ternary tree, and its properties were investigated in [5, 6, 7]. In [2], the first author has proved that any infinite finitely generated subgroup of Γ is abstractly commensurable with either Γ or $\Gamma \times \Gamma$. The question naturally arises whether there are two commensurability classes of infinite finitely generated subgroups or just one; the latter case arises if and only if Γ and $\Gamma \times \Gamma$ are abstractly commensurable. We shall use Theorem 1.2 to show that this is not the case.

Definition. For an infinite group H, let $b(H)$ be the largest number r such that H has r infinite normal subgroups that generate their direct product in H; if no such r exists write $b(H) = \infty$.

If H is a subgroup of finite index in a branch group (or if H is any infinite group having no non-trivial finite normal subgroups) then $b(H)$ is the maximal number of factors in an irredundant subdirect product decomposition of H.

Corollary 3.1. *Let G be a branch group acting on a tree T and let $H \leqslant_{\mathrm{f}} G$. Then $b(H)$ is finite and is the maximum number of H-orbits on any layer of T.*

Proof. By Theorem 1.1 and Remark 2.1, there exist n_0 and some $m \leqslant |G : H|$ such that $\mathrm{rist}_G(n_0)' \leqslant H$ and such that H has m orbits on L_n for all $n \geqslant n_0$. If $X_1, \ldots, X_m$ are the orbits of H on L_{n_0} then the m subgroups $\mathrm{rist}_G(X_i)'$ are infinite normal subgroups of H and generate their direct product.

Now let $K_1, \ldots, K_r$ be infinite normal subgroups of H that generate their direct product. By Theorem 1.2, for each i we can find an integer $n_i \geqslant n_0$ and a union X_i of H-orbits on L_{n_i} such that $\mathrm{rist}_G(X_i)' \leqslant K_i$. Let $n = \max\{n_1, \ldots, n_r\}$ and for each i choose an H-orbit Y_i of L_n consisting of descendants of elements of X_i. Obviously the sets Y_i are disjoint and so $r \leqslant m$. $\square$

Corollary 3.2. *Suppose that G is a branch group on the p-ary tree T and $H \leqslant G$, where p is a prime. If H acts on every layer of T as a p-group, then the number of H-orbits on each layer is congruent to $1 \mod p-1$. In particular, $b(H) \equiv 1 \mod p-1$ for every such H of finite index in G.*

Proof. The first assertion is elementary. Pick some layer L_n. Then the size of each H-orbit in L_n is a power of p. Let s_i denote the number of orbits of size p^i for $i = 0, \ldots, n$. Then $p^n = s_0 + ps_1 + \cdots + p^n s_n$, so modulo $p-1$ we obtain $1 \equiv s_0 + s_1 + \cdots + s_n$. $\square$

Using these corollaries we show that two direct products of branch groups on p-ary trees cannot be abstractly commensurable if the numbers of direct factors are not congruent mod $p-1$. The next lemma will be applied to branch groups and their finite index subgroups, but we present it in greater generality.

Lemma 3.3. *Let $\mathfrak{C}$ be a class of groups which have no non-trivial abelian normal subgroups and such that $b(G)$ is finite for all G in $\mathfrak{C}$. Let $H_1, \ldots, H_n$ be groups in $\mathfrak{C}$, and H a subgroup of finite index in the direct product $D = H_1 \times \cdots \times H_n$ such that each projection $\rho_j : H \to H_j$ is surjective. Then $b(H) = b(H_1) + \cdots + b(H_n)$.*

Proof. Let $K_1, \ldots, K_r$ be infinite normal subgroups of H that generate their direct product in H. Fix $i \in \{1, \ldots, r\}$ and for each $j \in \{1, \ldots, n\}$, write $K_{i,j} = \rho_j(K_i) \lhd H_j$. If $K_{i,j}$ is finite for all j then K_i is finite, a contradiction. So for each $i \in \{1, \ldots, r\}$ there is some j such that $K_{i,j}$ is infinite. Therefore $r \leqslant \sum r_j$, where for each j, we write r_j for the number of indices i with $K_{i,j}$ infinite. Now each subgroup K_i centralizes all other subgroups $K_{i'}$ with $i' \neq i$. Therefore for fixed j, each $K_{i,j}$ centralizes the product $P_{i,j}$ of all subgroups $K_{i',j}$ with $i' \neq i$; and since H_j has no non-trivial abelian normal subgroups, the normal subgroups $K_{i,j}, P_{i,j}$ intersect trivially. Thus the subgroups $K_{i,j}$ with $1 \leqslant i \leqslant r$ generate their direct product in H_j. It follows that $r_j \leqslant b(H_j)$, and hence $r \leqslant b(H_1) + \cdots + b(H_n)$. In particular, $b(H)$ is finite and bounded by $\sum b(H_j)$.

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It remains to prove that $\sum b(H_j) \leqslant b(H)$. For each j, find a family of $b(H_j)$ infinite normal subgroups of H_j that generate their direct product in H_j. If $L \lhd H_j$ is one of these subgroups, then its image $\bar{L}$ under the natural injection $H_j \to D$ is an infinite normal subgroup of D; hence $\bar{L} \cap H$ is an infinite normal subgroup of H. In this way we find $\sum b(H_j)$ infinite normal subgroups of H that evidently generate their direct product. The result follows. $\square$

Corollary 3.4. *Let $\mathfrak{D}$ be the family of branch groups which act as a p-group on each layer of the p-ary tree T. Let Γ_1 and Γ_2 be, respectively, direct products of n_1 and n_2 groups in $\mathfrak{D}$. If Γ_1 and Γ_2 are abstractly commensurable then $n_1 \equiv n_2$ mod $p - 1$.*

Proof. By Corollary 3.1, $\mathfrak{D}$ is contained in $\mathfrak{C}$. Thus, combining Corollary 3.2 and the previous result, for $i = 1, 2$ we have $b(H_i) \equiv n_i \mod p - 1$ for all $H_i \leqslant_{\mathrm{f}} \Gamma_i$. Hence if there are isomorphic such subgroups H_1, H_2 then we must have $n_1 \equiv n_2 \mod p - 1$. $\square$

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